

Two analogs of Pleijel's inequality

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1 Formulation of results

Let $N(\lambda)$ be a nondecreasing function defined on $\mathbb{R}_+ = (0, +\infty)$ such that $N(\lambda) = 0$ for small λ and

$$\int_0^\infty \lambda^{-1} dN(\lambda) < \infty. \quad (1)$$

The *Stieltjes transform* of $N(\lambda)$ is defined as *

$$S(\zeta) = \int_0^\infty (\lambda - \zeta)^{-1} dN(\lambda), \quad \zeta \notin \mathbb{R}_+. \quad (2)$$

Fix a point $\zeta_0 = \lambda_0 + i\eta_0$ in the first quadrant of the complex plane \mathbb{C} . Denote by Γ a contour that connects the point ζ_0 to $\bar{\zeta}_0 = \lambda_0 - i\eta_0$ and does not cross the integration path \mathbb{R}_+ of (2).

Åke Pleijel in [1] obtained the inequality

$$\left| N(\lambda_0) - \frac{1}{2\pi i} \int_\Gamma S(\zeta) d\zeta \right| \leq \eta_0 \sqrt{1 + \pi^{-2}} |S(\zeta_0)| \quad (3)$$

and used it to give a short proof of Malliavin's [2] Tauberian theorem. The inequality (3) found applications in spectral theory of differential and pseudo-differential operators (see e.g. [3, 4]). In the present paper two generalizations of this inequality are derived.

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*There exists an alternative convention according to which the Stieltjes transform of $f(t)$ is defined as $\int_0^\infty (\zeta + t)^{-1} f(t) dt$.

For $\alpha > 0$, the *Riesz mean of order α of $N(\lambda)$* is

$$N^{(\alpha)}(\lambda) = \int_0^\lambda \left(1 - \frac{x}{\lambda}\right)^\alpha dN(x), \quad \lambda > 0. \quad (4)$$

Given a power asymptotics of the Stieltjes transform of $N(\lambda)$ along a certain parabola-like curve in \mathbb{C} that avoids \mathbb{R}_+ , the asymptotics of the Riesz means as $\lambda \rightarrow +\infty$ can be recovered using the following theorem.

Theorem 1 *Let the function $N(\lambda)$ be constant in a neighbourhood of λ_0 . Then for any $\alpha > 0$*

$$\left| N^{(\alpha)}(\lambda_0) - \frac{1}{2\pi i} \int_\Gamma S(\zeta) \left(1 - \frac{\zeta}{\lambda_0}\right)^\alpha d\zeta \right| \leq \frac{1}{\alpha\pi} \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \eta_0 |S(\zeta_0)|. \quad (5)$$

For $\alpha < 1$ the factor $(\alpha\pi)^{-1}$ in the right-hand side may be replaced by $\sqrt{\pi^{-2} + 1/4}$. *

Henceforth the branch $z^\alpha = \exp(\alpha \ln z)$ with $-\pi < \operatorname{Im} \ln z \leq \pi$ is assumed.

If instead of (1) a weaker ** condition with some integer $q > 1$

$$\int_0^\infty \lambda^{-q} dN(\lambda) < \infty \quad (6)$$

holds, then the leading term of the asymptotics of $N(\lambda)$ can be recovered by means of the next theorem from the behaviour of its *generalized Stieltjes transform*

$$S_q(\zeta) = \int_0^\infty (\lambda - \zeta)^{-q} dN(\lambda), \quad \zeta \notin \mathbb{R}_+. \quad (7)$$

Theorem 2 *Let the function $N(\lambda)$ satisfy (6) and be constant in a neighbourhood of λ_0 . There exist constants C_0, C_1, \dots, C_{q-2} (which depend only on q) such that*

$$\begin{aligned} & \left| N(\lambda_0) - \frac{1}{2\pi i} \int_\Gamma S_q(\zeta) (\zeta - \lambda_0)^{q-1} d\zeta \right| \\ & \leq \sum_{m=0}^{q-2} C_m \eta_0^{q-1-m} \left| \int_\Gamma S_q(\zeta) (\lambda_0 - \zeta)^m d\zeta \right|. \end{aligned} \quad (8)$$

* (Added in translation.) A continuous dependence of the constant on α can be achieved by replacing the term $1/4$ with $(1 - \alpha^{1+\varepsilon})/4$, where $\varepsilon \geq \varepsilon_0 \approx 1/16$ (found numerically).

** Inequality (6) is weaker than (1) since we assume that $dN(\lambda) = 0$ near $\lambda = 0$.

2 Proof of Theorem 1

a. The left-hand side of (5) vanishes if one uses a closed contour of integration consisting of Γ and the segment $[\zeta, \bar{\zeta}_0]$. Indeed, since $N(\lambda)$ is assumed constant in the vicinity of λ_0 , one may change the order of integration:

$$\frac{1}{2\pi i} \oint S(\zeta) \left(1 - \frac{\zeta}{\lambda_0}\right)^\alpha d\zeta = \int_0^\infty dN(\lambda) \frac{1}{2\pi i} \oint \left(\frac{\lambda_0 - \zeta}{\lambda_0}\right)^\alpha \frac{d\zeta}{\lambda - \zeta}.$$

The inner integral in the r.h.s. equals $2\pi i(1 - \lambda/\lambda_0)^\alpha$ when $\lambda < \lambda_0$, and 0 when $\lambda > \lambda_0$. Thus the r.h.s. equals $N^{(\alpha)}(\lambda_0)$. In order to prove (5) we have to evaluate

$$R_\alpha(\zeta_0) = \frac{1}{2\pi i} \int_{\bar{\zeta}_0}^{\zeta_0} S(\zeta) \left(1 - \frac{\zeta}{\lambda_0}\right)^\alpha d\zeta \quad (9)$$

(where the integration path is the vertical segment).

Set $\zeta = \lambda_0 + i|\lambda - \lambda_0|\tau$, $s(\lambda) = \text{sgn}(\lambda - \lambda_0)$, $u(\lambda) = \eta_0|\lambda - \lambda_0|^{-1}$. Changing the order of integration in (9), we get

$$R_\alpha(\zeta_0) = \frac{(-i)^\alpha}{2\pi} \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \int_0^\infty \frac{dN(\lambda)}{u^\alpha(\lambda)} \int_{-u(\lambda)}^{u(\lambda)} \frac{\tau^\alpha (s(\lambda) + i\tau)}{1 + \tau^2} d\tau. \quad (10)$$

We will find constants $c_1 \in \mathbb{C}$ and $c_2 > 0$ so that for any $u > 0$ and $s = \pm 1$ the inequality

$$\left| \frac{1}{u^\alpha} \int_{-u}^u \frac{\tau^\alpha (s + i\tau)}{1 + \tau^2} d\tau - c_1 \frac{su}{1 + u^2} \right| \leq c_2 \frac{u^2}{1 + u^2} \quad (11)$$

will hold. Having (11) and the identities

$$\int_0^\infty \frac{u(\lambda)s(\lambda)}{1 + u^2(\lambda)} dN(\lambda) = \eta_0 \text{Re } S(\zeta_0)$$

and

$$\int_0^\infty \frac{u^2(\lambda)}{1 + u^2(\lambda)} dN(\lambda) = \eta_0 \text{Im } S(\zeta_0)$$

one can estimate the r.h.s. in (10) by means of the Schwarz inequality:

$$\begin{aligned} |R_\alpha(\zeta_0)| &\leq \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \frac{\eta_0}{2\pi} \sqrt{|c_1|^2 + c_2^2} \sqrt{\operatorname{Re}^2 S(\zeta_0) + \operatorname{Im}^2 S(\zeta_0)} \\ &= \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \frac{\eta_0}{2\pi} c_3 |S(\zeta_0)|. \end{aligned} \quad (12)$$

Our task is thus reduced to establishing the inequality (11) with constants c_1, c_2 such that $\sqrt{|c_1|^2 + c_2^2} \leq 2\alpha^{-1}$ (and $\leq \sqrt{\pi^2 + 4}$ if $\alpha < 1$).

b. Using the change of variable $\tau \mapsto -\tau$ on $[-u, 0]$, we get

$$\int_{-u}^u \frac{\tau^\alpha (s + i\tau)}{1 + \tau^2} d\tau = 2e^{\frac{i\pi\alpha}{2}} \int_0^u \frac{(a\tau^{\alpha+1} + sb\tau^\alpha)(1 + \tau^2)}{(1 + \tau^2)^2} d\tau \quad (13)$$

where

$$a = \sin \frac{\pi\alpha}{2}, \quad b = \cos \frac{\pi\alpha}{2}. \quad (14)$$

We will need the integral representations

$$\frac{u^{\alpha+1}}{1 + u^2} = \frac{\tau^{\alpha+1}}{1 + \tau^2} \Big|_0^u = \int_0^u \frac{(\alpha - 1)\tau^{\alpha+2} + (\alpha + 1)\tau^\alpha}{(1 + \tau^2)^2} d\tau \quad (15)$$

and

$$\frac{u^{\alpha+2}}{1 + u^2} = \int_0^u \frac{\alpha\tau^{\alpha+3} + (\alpha + 2)\tau^{\alpha+1}}{(1 + \tau^2)^2} d\tau. \quad (16)$$

Multiplying the inequality (11) by u^α and making the substitutions (13), (15), (16), we transform (11) to the equivalent form

$$\begin{aligned} &\left| \int_0^u \frac{a\tau^{\alpha+3} + sk_- \tau^{\alpha+2} + a\tau^{\alpha+1} + sk_+ \tau^\alpha}{(1 + \tau^2)^2} d\tau \right| \\ &\leq \frac{1}{2} c_2 \int_0^u \frac{\alpha\tau^{\alpha+3} + (\alpha + 2)\tau^{\alpha+1}}{(1 + \tau^2)^2} d\tau, \end{aligned} \quad (17)$$

where

$$k_\pm = b - \frac{c_1}{2} e^{-i\pi\alpha/2} (\alpha \pm 1).$$

In order for (17) to hold for small positive u , the coefficient of τ^α must equal 0, so we set

$$c_1 = \frac{2b}{\alpha + 1} e^{i\pi\alpha/2}. \quad (18)$$

With this value of c_1 , the numerator in the l.h.s. of (17) is real. Clearing the absolute value notation, we rewrite (17) as a system of two inequalities, in which we leave the least favorable sign of s (so as to make the coefficient of $\tau^{\alpha+2}$ negative):

$$\int_0^u \frac{(c_2\alpha \mp 2a)\tau^{\alpha+3} - \frac{4|b|}{\alpha+1}\tau^{\alpha+2} + (c_2\alpha + 2c_2 \mp 2a)\tau^{\alpha+1}}{(1+\tau^2)^2} d\tau \geq 0.$$

Taking the least favorable sign in front of $2a$, we get

$$\int_0^u \frac{P_2(\tau) \tau^{\alpha+1}}{(1+\tau^2)^2} d\tau \geq 0, \quad (19)$$

where

$$P_2(\tau) = (c_2\alpha - 2|a|)\tau^2 - \frac{4|b|}{\alpha+1}\tau + (c_2\alpha - 2|a| + 2c_2).$$

The inequality (19) with c_1 defined by (18) implies (11). The rest of the proof amounts to finding an appropriate value of c_2 .

c. It suffices to ensure that the quadratic polynomial $P_2(\tau)$ is nonnegative. Let us choose the value of c_2 that makes its discriminant equal to zero:

$$c_2 = 2 \frac{|a|(\alpha+1)^2 + \sqrt{a^2 + \alpha(\alpha+2)}}{\alpha(\alpha+1)(\alpha+2)}. \quad (20)$$

In view of (14) we have $|a|, |b| \leq 1$, $a^2 + b^2 = 1$, and the following estimates readily follow:

$$\frac{|a|}{\alpha} \leq \frac{c_2}{2} \leq \frac{|a|(\alpha+1) + 1}{\alpha(\alpha+2)}. \quad (21)$$

The left estimate shows that the coefficient of τ^2 in $P_2(\tau)$ is nonnegative. By our choice of c_2 , this leads to (19) and hence to (11).

Using the right inequality in (21) together with (14) and (18), we find $|c_1|^2 + c_2^2 \leq 4\alpha^{-2}$, as required.

d. To finish the proof, let us show that if $\alpha < 1$ then one may use the value $c_2 = 2|a|\alpha^{-1}$ instead of (20). With this new choice of c_2 , the constant c_3 in the r.h.s. of (12) becomes

$$c_3 = 2\sqrt{\left(\frac{\cos(\pi\alpha/2)}{\alpha+1}\right)^2 + \left(\frac{\sin(\pi\alpha/2)}{\alpha}\right)^2} \leq \sqrt{4 + \pi^2},$$

as the theorem claims. We have to verify the inequality

$$4 \frac{|a|}{\alpha} \int_0^u \frac{\tau^{\alpha+1}}{(1+\tau^2)^2} d\tau - 4 \frac{|b|}{\alpha+1} \int_0^u \frac{\tau^{\alpha+2}}{(1+\tau^2)^2} d\tau \geq 0$$

in the interval $0 < \alpha < 1$. The left-hand side, as a function of u , is positive for small u and has a unique critical point (maximum) on \mathbb{R}_+ . It remains to check that

$$\frac{|a|}{\alpha} \int_0^\infty \frac{\tau^{\alpha+1}}{(1+\tau^2)^2} d\tau \geq \frac{|b|}{\alpha+1} \int_0^\infty \frac{\tau^{\alpha+2}}{(1+\tau^2)^2} d\tau.$$

Using the substitution $\tau^2 = t$ we express the integrals in terms of Euler's Beta function and the last inequality takes the form

$$\frac{|a|}{\alpha} B\left(\frac{\alpha+2}{2}, \frac{2-\alpha}{2}\right) \geq \frac{|b|}{\alpha+1} B\left(\frac{\alpha+3}{2}, \frac{1-\alpha}{2}\right).$$

The right and left sides are in fact equal: it follows from (14) and the identities

$$B\left(\frac{\alpha+2}{2}, \frac{2-\alpha}{2}\right) = \frac{\pi\alpha/2}{\sin(\pi\alpha/2)}, \quad B\left(\frac{\alpha+3}{2}, \frac{1-\alpha}{2}\right) = \frac{\pi(\alpha+1)/2}{\cos(\pi\alpha/2)}.$$

The proof is complete.

3 Proof of Theorem 2

a. The left-hand side of the inequality (8), likewise the l.h.s. of the inequality (5) in Theorem 1, vanishes if the closed contour of integration consisting of Γ and the segment $[\zeta_0, \bar{\zeta}_0]$ is used. In the right-hand side of (8), integration over Γ can be replaced by integration over $[\bar{\zeta}_0, \zeta_0]$ since

$$\int_\Gamma S_q(\zeta)(\lambda_0 - \zeta)^m d\zeta = \int_0^\infty dN(\lambda) \int_\Gamma \frac{(\lambda_0 - \zeta)^m}{(\lambda - \zeta)^q} d\zeta,$$

and the residue of the integrand at $\zeta = \lambda$ equals 0 as $m \leq q-2$. Therefore (8) is equivalent to the inequality

$$\left| \int_0^\infty V_{q,q-1}(\lambda) dN(\lambda) \right| \leq \sum_{m=0}^{q-2} C_m \left| \int_0^\infty V_{q,m}(\lambda) dN(\lambda) \right|, \quad (22)$$

where

$$V_{q,m}(\lambda) = \eta_0^{q-1-m} \int_{\bar{\zeta}_0}^{\zeta_0} \frac{(\lambda_0 - \zeta)^m}{(\lambda - \zeta)^q} d\zeta, \quad m = 0, 1, \dots, q-1. \quad (23)$$

The substitutions $\zeta = \lambda_0 + i\eta_0\tau$, $\mu = (\lambda - \lambda_0)\eta_0^{-1}$ bring (23) to the form

$$V_{q,m}(\lambda) = (-i)^{m-1} T_{q,m}(\mu), \quad (24)$$

where

$$T_{q,m}(\mu) = \int_{-1}^1 \frac{\tau^m d\tau}{(\mu - i\tau)^q}. \quad (25)$$

b. Let us study properties of the functions $T_{q,m}(\mu)$.

1°. The function $T_{q,m}(\mu)$ is even if $q-m$ is even, and odd if $q-m$ is odd. This is verified by changing μ into $-\mu$ in (25) and τ into $-\tau$.

2°. If $0 \leq m \leq q-2$, then we can write

$$T_{q,m}(\mu) = \frac{P_{q,m}(\mu)}{(\mu^2 + 1)^{q-1}}, \quad (26)$$

where $P_{q,m}(\mu)$ is a polynomial (even or odd depending on the evenness of $q-m$). Indeed, expanding τ^m in powers of $\mu - i\tau$, integrating the resulting linear combination of the functions $(\mu - i\tau)^{n-q}$ ($n = 0, \dots, m$) with respect to τ , and taking the common denominator, we obtain (26).

3°. From (25) it is easy to find the asymptotics of $T_{q,m}$ as $\mu \rightarrow +\infty$:

$$\begin{aligned} T_{q,m}(\mu) &= b_{q,m} \mu^{-q} + O(\mu^{-q-2}) && \text{for } m \text{ even,} \\ T_{q,m}(\mu) &= b_{q,m} \mu^{-q-1} && \text{for } m \text{ odd,} \end{aligned} \quad (27)$$

where $b_{q,m} \neq 0$.

Comparing to (26), we see that the polynomial $P_{q,m}$ ($m \leq q-2$) is of exact degree $q-2$ for m even, and of exact degree $q-3$ for m odd.

c. Let us show that $\{P_{q,m}\}$, $m = 0, \dots, q-2$, is a basis in the space of polynomials of degree at most $q-2$. It suffices to verify that the corresponding functions $T_{q,m}$ are linearly independent. Suppose, to the contrary, that some their linear combination is zero:

$$L(\mu) = \int_{-1}^1 \frac{U(\tau)}{(\mu - i\tau)^q} d\tau = 0,$$

where $U(\tau)$ is a polynomial of degree at most $q - 2$. Consider L as an analytic function of complex variable μ . It is regular outside the segment $[-i, i]$, therefore, it equals zero identically. Let γ be a closed contour around the segment $[-i, i]$. For all integer $n \geq q - 1$ we have

$$0 = \int_{\gamma} L(z) z^n dz = \int_{-1}^1 U(\tau) d\tau \int_{\gamma} \frac{z^n dz}{(z - i\tau)^q} = \beta_n \int_{-1}^1 U(\tau) \tau^{n-q+1} d\tau,$$

and $\beta_n \neq 0$. Hence for any polynomial $\tilde{U}(\tau)$

$$\int_{-1}^1 U(\tau) \tilde{U}(\tau) d\tau = 0.$$

Taking \tilde{U} to be the complex-conjugate of U leads to the conclusion $U \equiv 0$, which proves the linear independence of the functions $T_{q,m}$.

d. Consider first the case of even q . The result of **(c)** shows that for some C'_0, \dots, C'_{q-2}

$$\sum_{m=0}^{q-2} C'_m P_{q,m}(\mu) = 1 + \mu^{q-2},$$

or — cf. (26) — that

$$\sum_{m=0}^{q-2} C'_m T_{q,m}(\mu) = \frac{1 + \mu^{q-2}}{(1 + \mu^2)^{q-1}} =: H_q(\mu). \quad (28)$$

As $|\mu| \rightarrow \infty$, we have

$$|T_{q,q-1}(\mu)| \sim |b_{q,q-1}| |\mu|^{-q-1} = o(H_q(\mu)).$$

The function $T_{q,q-1}(\mu)$ is bounded, while $\min H_q(\mu) > 0$ on every finite interval. Therefore there exists a positive C such that $|T_{q,q-1}(\mu)| \leq C H_q(\mu)$ for all real $\mu \neq 0$. Using (24) and passing from $T_{q,m}$ to $V_{q,m}$, then integrating with $dN(\lambda)$, we get (22).

e. Now consider the case of odd q . There exist constants C''_0, \dots, C''_{q-3} such that

$$\sum_{m=0}^{q-3} C''_m T_{q,m}(\mu) = \frac{1 + \mu^{q-3}}{(1 + \mu^2)^{q-1}} =: H_q(\mu). \quad (29)$$

Now $T_{q,q-1}(\mu)$ decays at infinity slower than $H_q(\mu)$. However, as seen from (27),

$$\left| T_{q,q-1}(\mu) - \frac{b_{q,q-1}}{b_{q,0}} T_{q,0}(\mu) \right| = O(\mu^{-q-2}) = o(H_q(\mu)), \quad |\mu| \rightarrow \infty.$$

Here, like previously in (d), the left-hand side is a bounded function, while $\min H_q(\mu) > 0$ on every finite interval. Therefore there exists a positive C such that for all real $\mu \neq 0$.

$$\left| T_{q,q-1}(\mu) - \frac{b_{q,q-1}}{b_{q,0}} T_{q,0}(\mu) \right| \leq C \sum_{m=0}^{q-3} C_m'' T_{q,m}(\mu).$$

Passing from $T_{q,m}$ to $V_{q,m}$, integrating with $dN(\lambda)$, and bringing the term $\text{const} \left| \int_0^\infty V_{q,0}(\lambda) dN(\lambda) \right|$ over to the right-hand side, we get (22).

The proof is finished.

Remark. Note that due to (b1°) the coefficients of odd functions $T_{q,m}$ in (28), (29) are equal to 0. Hence in the right-hand side of (8) the actual summation is carried over the values of index m for which $q - m$ is even; if q is odd, the value $m = 0$ is also included.

In conclusion I would like to thank M.S. Agranovich for suggesting the problem and for attention to this work.

References

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Remarks added in translation

1. In both Theorems 1 and 2, the function $N(\lambda)$ does not have to be continuous at λ_0 . If λ_0 is a point of discontinuity of $N(\lambda)$, Theorem 1 remains valid without change, while in Theorem 2 the value $N(\lambda_0)$ in the left-hand side can be replaced by any value between $N(\lambda_0 - 0)$ and $N(\lambda_0 + 0)$.

2. The following theorem stands in the same relation to Theorem 2 as Theorem 1 to Pleijel's inequality (3).

Theorem 3 *Let function $N(\lambda)$ defined on \mathbb{R}_+ be nondecreasing, equal zero near $\lambda = 0$, and satisfy condition (6). For any $\alpha > 0$ and any integer $q = 2, 3, \dots$ there exist nonnegative constants C_0, \dots, C_{q-2} , depending on q and α , such that for any $\lambda > 0$*

$$\begin{aligned} & \left| N^\alpha(\lambda_0) - \frac{\alpha B(q, \alpha)}{2\pi i} \int_{\Gamma} S_q(\zeta) (\zeta - \lambda_0)^{q-1} \left(1 - \frac{\zeta}{\lambda_0}\right)^\alpha d\zeta \right| \\ & \leq \sum_{m=0}^{q-2} C_m \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \cdot \eta_0^{q-1-m} \left| \int_{\Gamma} S_q(\zeta) (\zeta - \lambda_0)^m d\zeta \right|, \end{aligned}$$

where

$$B(q, \alpha) = \frac{\Gamma(q)\Gamma(\alpha)}{\Gamma(q + \alpha)}.$$

Proof: Repeat the proof of Theorem 2 replacing the function $T_{q,q-1}(\mu)$ by the function

$$T_{q,q-1+\alpha}(\mu) = \int_{-1}^1 \frac{\tau^{q-1+\alpha}}{(\mu - i\tau)^q} d\tau.$$

3. An application of Theorem 1 can be found in:

Agranovich M.S. Elliptic operators on closed manifolds. Encycl. Math. Sci., 63 (Partial Differential Equations VI), Springer-Verlag, 1994, Theorem 6.1.6.